1.4. Transform-Domain Representation of Discrete Signals and LTI Systems

1.4.1. Z -Transform

Definition: The *Z* – transform of a discrete-time signal *x(n)* is defined as the power series:

$$X(z) = \sum_{k=-\infty}^{\infty} x(n) z^{-k} \qquad X(z) = Z[x(n)]$$

where *z* is a complex variable. The above given relations are sometimes called **the direct *Z* - transform** because they transform the time-domain signal *x(n)* into its complex-plane representation *X(z)*.

Since *Z* – transform is an infinite power series, it exists only for those values of *z* for which this series converges. The **region of convergence** of *X(z)* is the set of all values of *z* for which *X(z)* attains a finite value.

The procedure for transforming from z – domain to the time-domain is called **the inverse *Z* – transform**. It can be shown that the inverse *Z* – transform is given by

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \qquad x(n) = Z^{-1}\left[X(z)\right]$$

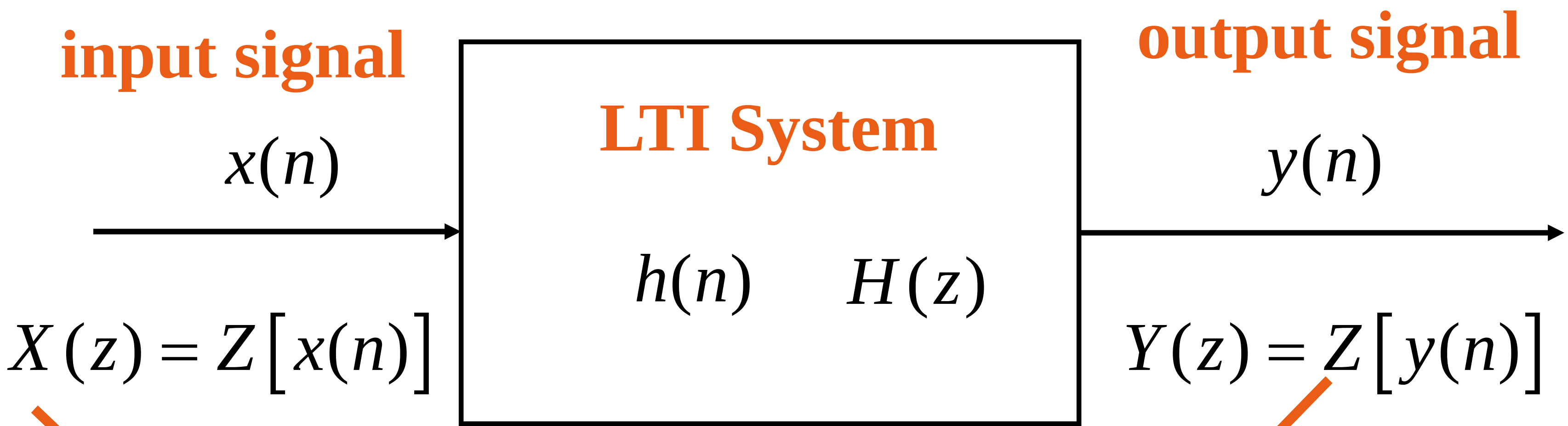
where *C* denotes the closed contour in the region of convergence of *X(z)* that encircles the origin.

1.4.2. Transfer Function

The LTI system can be described by means of **a constant coefficient linear difference equation** as follows

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

Application of the Z-transform to this equation under zero initial conditions leads to the notion of **a transfer function**.



$$H(z) = Z\left[h(n)\right]$$

Transfer function: the ratio of the Z - transform of the output signal and the Z - transform of the input signal of the LTI system:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Z[y(n)]}{Z[x(n)]}$$

LTI system: the Z-transform of the constant coefficient linear difference equation under zero initial conditions:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

$$Y(z) = \sum_{k=0}^{N} b(k)z^{-k}X(z) - \sum_{k=1}^{M} a(k)z^{-k}Y(z)$$

The transfer function of the LTI system:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1 + \sum_{k=1}^{M} a(k)z^{-k}}$$

H(z): may be viewed as a rational function of a complex variable z (z^{-1}).

1.4.3. Poles, Zeros, Pole-Zero Plot

Let us assume that *H(z)* has been expressed in its irreducible or so-called factorized form:

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}} = \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{N} (z - z_k)}{\prod_{k=1}^{M} (z - p_k)}$$

Zeros of *H(z)*: the set $\{z_k\}$ of *z*-plane for which $H(z_k)=0$

Poles of *H(z)*: the set $\{p_k\}$ of z -plane for which $H(p_k) \to \infty$

Pole-zero plot: the plot of **the zeros** and **the poles** of *H(z)* in the z-plane represents a strong tool for LTI system description.

Example: the 4-th order Butterworth low-pass filter, cut off frequency $\omega_1 = \pi/3$.

b =[0.0186 0.0743 0.1114 0.0743 0.0186]

a =[1.0000 -1.5704 1.2756 -0.4844 0.0762]

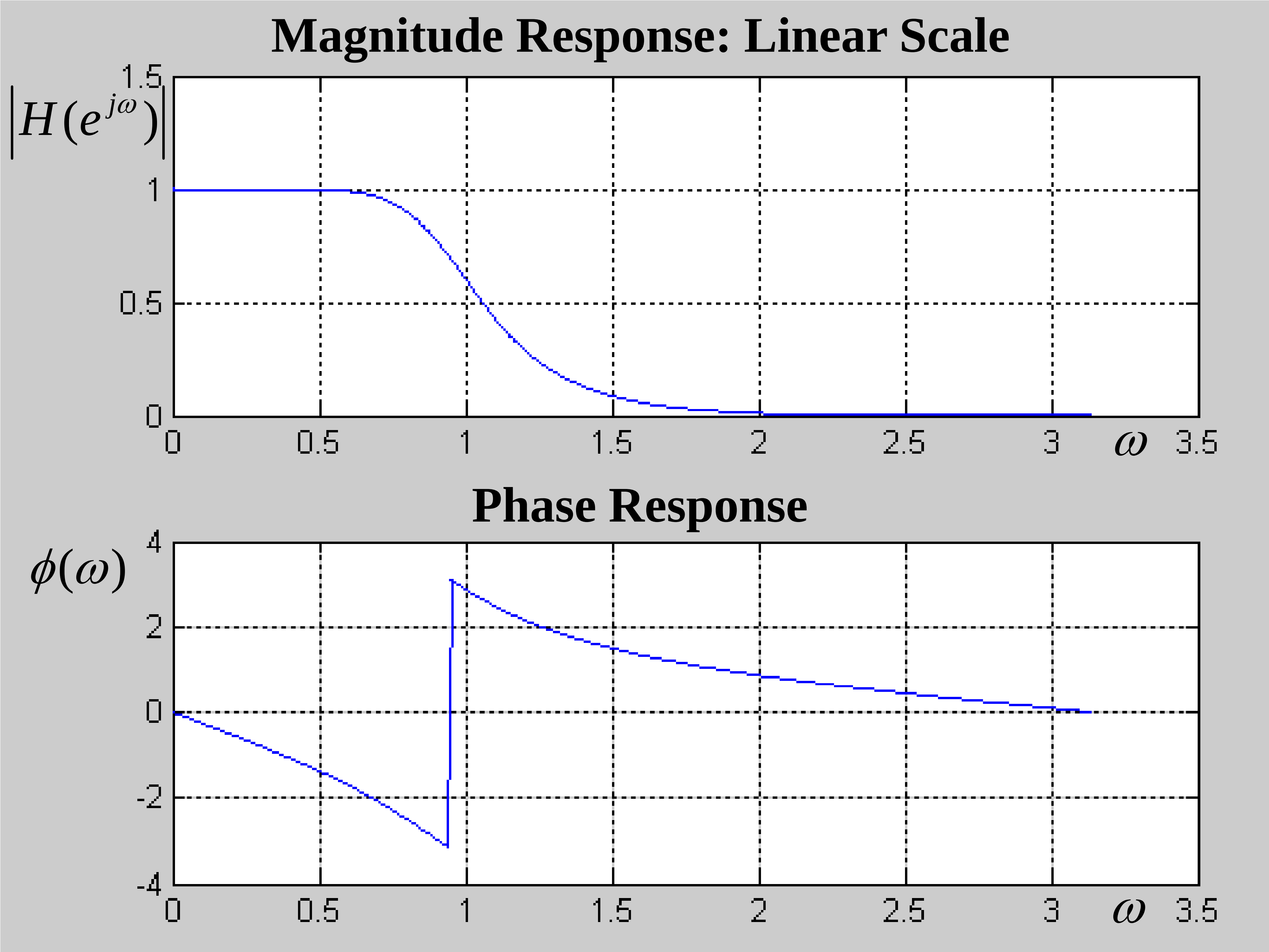
z_1= -1.0002, z_2= -1.0000 + 0.0002j

z_3= -1.0000 - 0.0002j, z_4= -0.9998

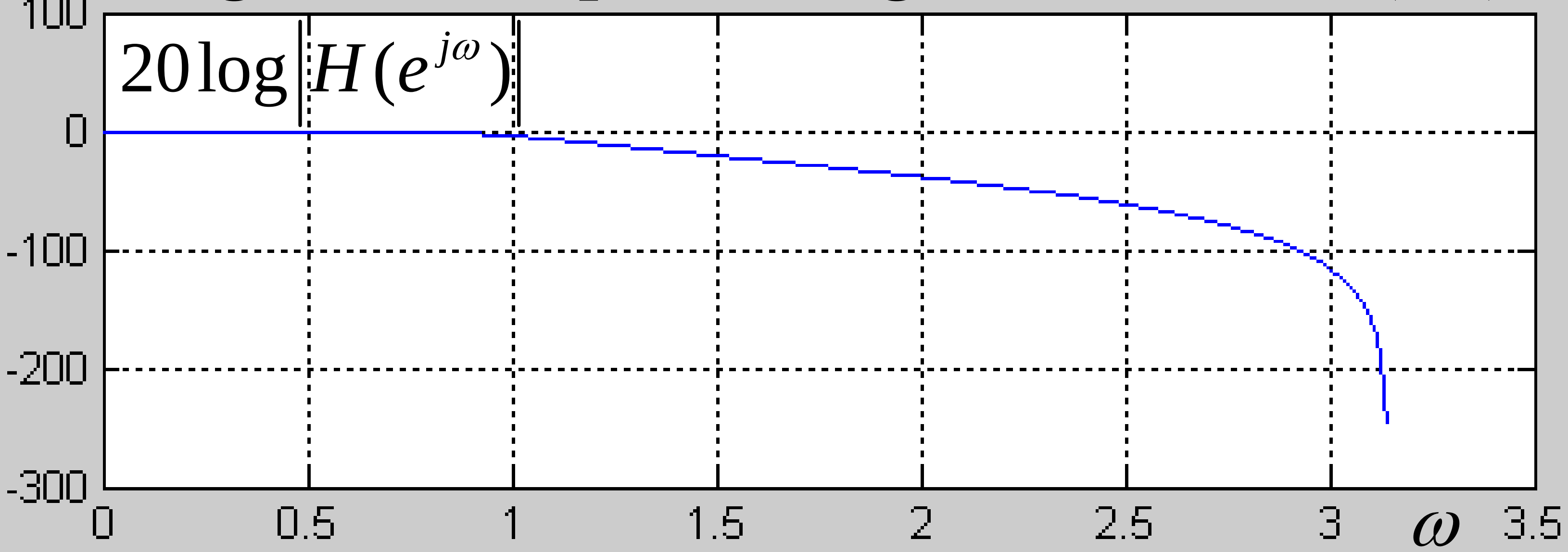
$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}} \qquad H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}}$$

p_1= 0.4488 + 0.5707j, p_2= 0.4488 - 0.5707j

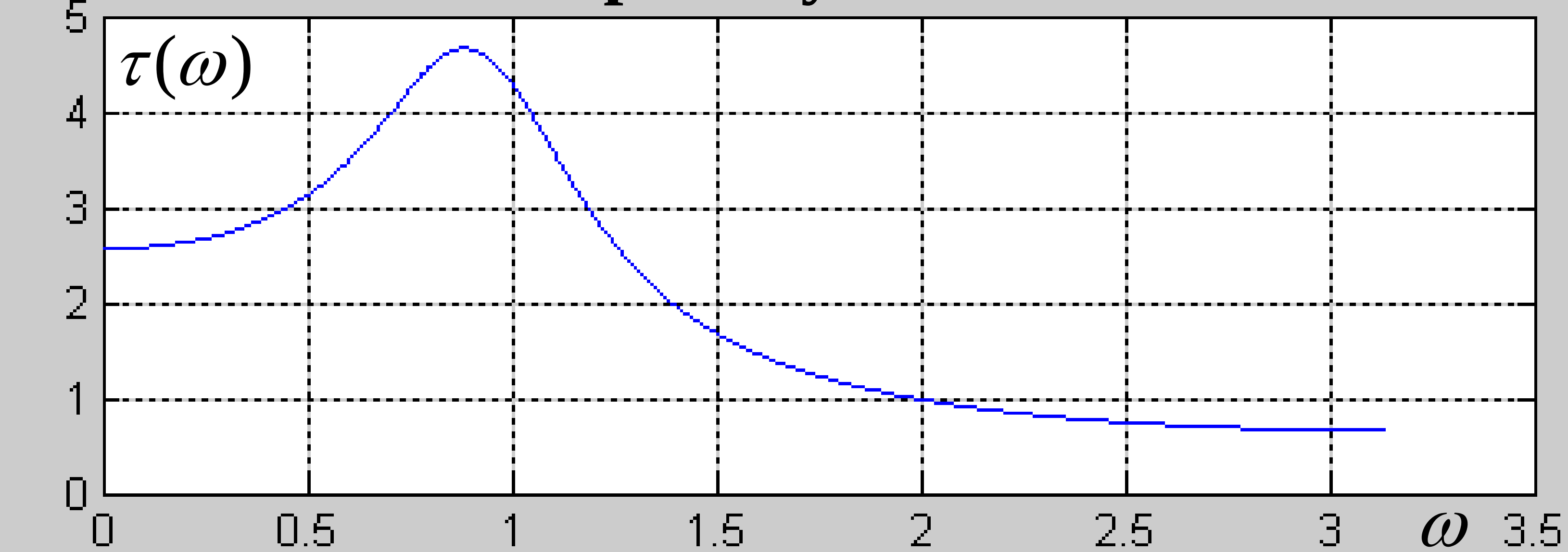
p_3= 0.3364 + 0.1772j, p_4= 0.3364 - 0.1772j

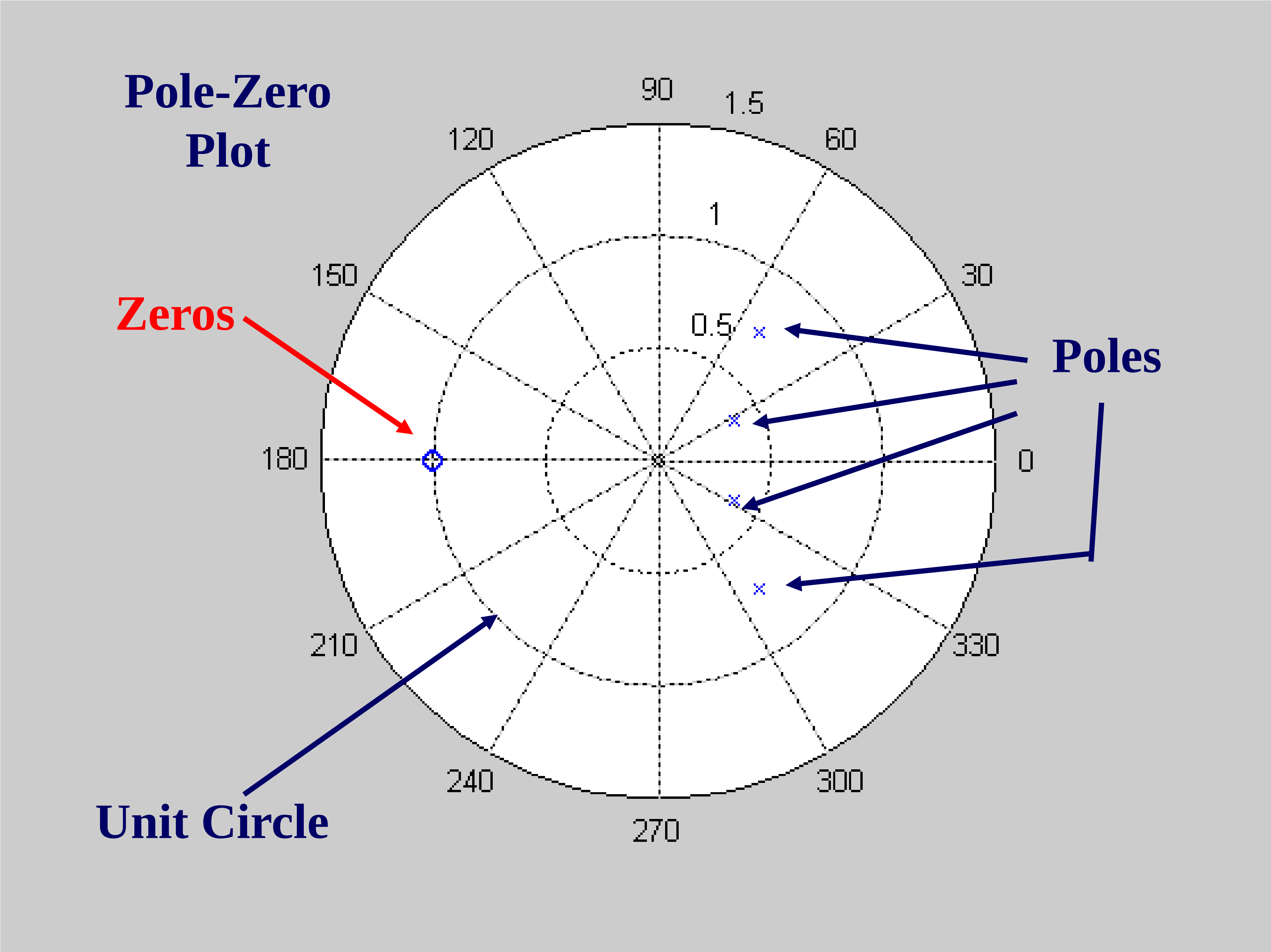

Magnitude Response: Linear Scale
$|H(e^{j\omega})|$
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
ω
3.5
Phase Response
$\phi(\omega)$
4
2
0
-2
-4
0
0.5
1
1.5
2
2.5
3
ω
3.5

Magnitude Response: Logarithmic Scale (dB)



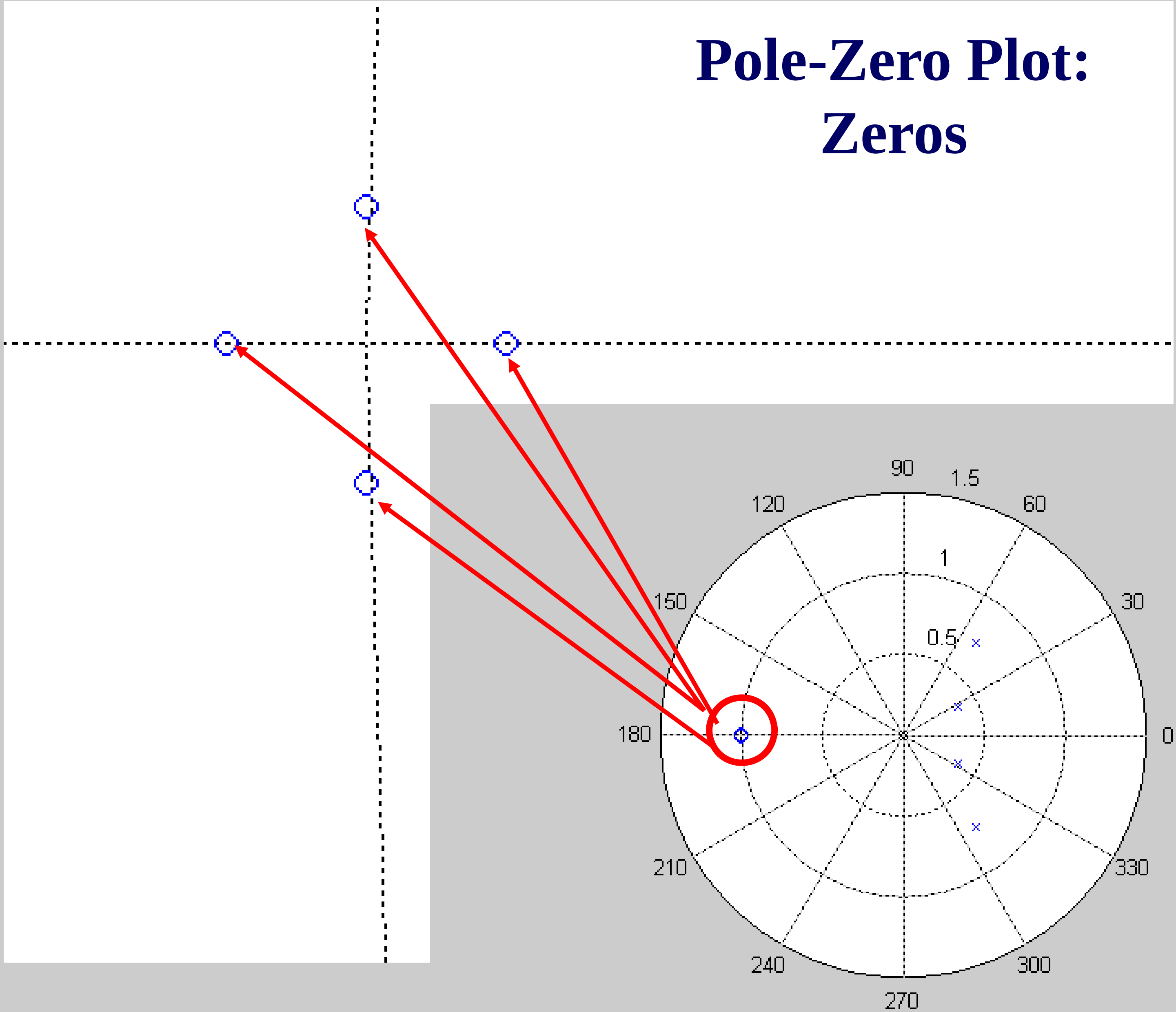
Group Delay Function





Pole-Zero
Plot
Zeros
Poles
Unit Circle
90
1.5
120
60
1
150
30
0.5
180
0
210
330
240
300
270

Pole-Zero Plot: Zeros



1.4.4. Transfer Function and Stability of LTI Systems

Condition: LTI system is BIBO stable if and only if the unit circle falls within the region of convergence of the power series expansion for its transfer function. In the case when the transfer function characterizes a causal LTI system, the stability condition is equivalent to the requirement that **the transfer function *H(z)* has all of its poles inside the unit circle.**

Example 1: stable system

$$H(z) = \frac{1 - 0.9z^{-1} + 0.18z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

$z_1 = 0.3 \quad p_1 = 0.4000 + 0.6928j \quad |p_1| = 0.8 < 1$

$z_2 = 0.6 \quad p_2 = 0.4000 - 0.6928j \quad |p_2| = 0.8 < 1$

Example 2: unstable system

$$H(z) = \frac{1 - 0.16z^{-2}}{1 - 1.1z^{-1} + 1.21z^{-2}}$$

$z_1 = 0.4 \quad p_1 = 0.5500 + 0.9526j \quad |p_1| = 1.1 > 1$

$z_2 = -0.4 \quad p_2 = 0.5500 - 0.9526j \quad |p_2| = 1.1 > 1$

1.4.5. LTI System Description. Summary

Time – Domain:

constant coefficient linear difference equation

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

$h(n)$

Z – Domain:

transfer function

Z / **Z^{-1}**

$$H(z) = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1 + \sum_{k=1}^{M} a(k)z^{-k}}$$

Frequency – Domain:

frequency response

FT / **FT^{-1}**

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{N} b(k)e^{-j\omega k}}{1 + \sum_{k=1}^{M} a(k)e^{-j\omega k}}$$

$z = e^{j\omega}$ $\quad e^{j\omega} = z$

Time – Domain: impulse response $h(k)$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \qquad H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$

Z **– Domain:** transfer function $H(z)$

$$H\left(e^{j\omega}\right) = H\left(z\right)_{z=e^{jw}} \qquad h(n) = \frac{1}{2\pi j} \oint_C H(z) z^{n-1} dz$$

Frequency – Domain: frequency response $H\left(e^{j\omega}\right)$

$$H(z) = H\left(e^{j\omega}\right)_{e^{j\omega}=z} \qquad h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega k} d\omega$$